Integration of Differential Equations

Consider $\qquad u' = f(u,t) \qquad$ (O.D.E.)

integrating from time $t=a$ to $t=b$

$$u(b) = u(a) + \int_{t=a}^{b} f(u,t)\,dt \qquad \text{(I.E.)}$$

Notice that to use the I.E. we need $f(u,t)$ over the interval $a<t<b$
$\Rightarrow$ we need u over $a<t<b$
(but u is what we're trying to find).

Let $\quad u_n = u(t_n) \qquad h = t_{n+1} - t_n$ (time step)

$u_0 = u(a)$ can be specified

but $u_n = u(a+nh)$ will be an approximation

could use $\quad u_0' = \dfrac{u_1 - u_0}{h} \quad$ Forward difference approximation for u' at t.

thus we get the explicit formula

$$u_1 = u_0 + h u_0'$$

$$\boxed{u_1 = u_0 + h\, f(u_0, t_0)}$$

Forward Euler

alternatively we could use

$$u_1' = \frac{u_1 - u_0}{h}$$ backward difference approximation at t_1

we get an implicit formula

$$u_1 = u_0 + h\, u_1'$$

$$\boxed{u_1 = u_0 + h\, f(u_1, t_1)}$$

Backward Euler

Notice that this is an implicit equation for u_1 and if f is non-linear we'll have to use a non-linear equation solver.

We could also use an average

$$\frac{u_1' + u_0'}{2} = \frac{u_1 - u_0}{h}$$

solving for u_1: $u_1 = u_0 + \frac{h}{2}(u_1' + u_0')$

$$\boxed{u_1 = u_0 + \frac{h}{2}\left(f(u_1, t_1) + f(u_0, t_0)\right)}$$

Trapezoidal rule.

Summary for arbitrary time step:

(F.E.) $u_{n+1} = u_n + h f(u_n, t_n)$

(B.E.) $u_{n+1} = u_n + h f(u_{n+1}, t_{n+1})$

(T.R.) $u_{n+1} = u_n + \frac{h}{2}\left(f(u_{n+1}, t_{n+1}) + f(u_n, t_n)\right)$

To use B.E. or T.R. we can use F.E. as a "predictor" and then iterate using

iteration no.

$$u'^{(i)}_{n+1} = f(u^{(i-1)}_{n+1}, t_{n+1}) \qquad i = 0, \ldots$$

$$u^{(0)}_{n+1} = u^{p}_{n+1} \qquad u^{p}_{n+1} = u_n + h f(u_n, t_n)$$

Scheme becomes:

"predictor" $u^{p}_{n+1} = u_n + h f(u_n, t_n)$

$u'^{(1)}_{n+1} = f(u^{p}_{n+1}, t_{n+1})$ (iterate)

"corrector" $u^{(1)}_{n+1} = u_n + h u'^{(1)}_{n+1}$ (B.E.)

$$u'^{(2)}_{n+1} = f(u^{(1)}_{n+1}, t_{n+1})$$

$$u^{(2)}_{n+1} = u_n + h u'^{(2)}_{n+1}$$

$$\vdots$$

until $u^{(i)}_{n+1}$ doesn't change any longer

Now suppose we have a linear system:

$$\underline{u}' = A\underline{u} + \underline{w}$$

$\underline{w}$ is a source vector

Then applying the B.E. Formula:

$$\underline{u}_{n+1} = \underline{u}_n + h\,\underline{u}'_{n+1}$$

$$\underline{u}_{n+1} = \underline{u}_n + h\left(A\underline{u}_{n+1} + \underline{w}_{n+1}\right)$$

solving for $\underline{u}_{n+1}$:

$$(1 - hA)\,\underline{u}_{n+1} = \underline{u}_n + h\,\underline{w}_{n+1}$$

we could invert $(1 - hA) = B$

but its better to decompose B into

the LU Factors $\quad B = LU$

Consider $\quad A\underline{x} = \underline{b} \qquad A = LU$

$$LU\underline{x} = \underline{b}$$

let $\quad \underline{z} = U\underline{x}$

$L\underline{z} = \underline{b} \implies$ solved by

$\underline{z} = L^{-1}\underline{b} \quad$ Forward substitution

then $\underline{x} = U^{-1}\underline{z} \implies$ solved by backward substitution

Notice that for linear systems, no iteration is required.

For the trapezoidal rule we have

$$\underline{u}_{n+1} = \underline{u}_n + \frac{h}{2}\left(A\underline{u}_{n+1} + \underline{w}_{n+1} + A\underline{u}_n + \underline{u}_n\right)$$

or

$$\boxed{\left(1 - \frac{h}{2}A\right)\underline{u}_{n+1} = \left(1 + \frac{h}{2}A\right)\underline{u}_n + \frac{h}{2}\left(\underline{w}_{n+1} + \underline{w}_n\right)}$$

Order of Integration (Truncation Error)

Consider the following general formula to approximate derivatives:

$$b_1 u_1' + b_0 u_0' = \frac{a_1 u_1 + a_0 u_0}{h}$$

$$a_1 u_1 + a_0 u_0 - h\left(b_1 u_1' + b_0 u_0'\right) = 0$$

(This is called a linear multistep approximation)

Now we can use the Taylor's expansions:

$$u_1 = u_0 + h u_0' + \frac{h^2}{2!} u_0'' + \frac{h^3}{3!} u_0''' + \ldots$$

$$u_1' = u_0' + h u_0'' + \frac{h^2}{2!} u_0''' + \frac{h^3}{3!} u_0^{iv} + \ldots$$

plugging into the LMS formula:

$$a_1\left[u_0 + h u_0' + \frac{h^2}{2!} u_0'' + \frac{h^3}{3!} u_0''' + \dots\right] + a_0 u_0$$
$$- h\, b_1\left[u_0' + h u_0'' + \frac{h^2}{2!} u_0''' + \frac{h^3}{3!} u_0^{iv} + \dots\right] - h b_0 u_0' = 0$$

$$u_0\left[a_1 + a_0\right] + h u_0'\left[a_1 - b_1 - b_0\right] = -h^2 u_0''\left[\frac{a_1}{2!} - b_1\right]$$
$$- h^3 u_0'''\left[\frac{a_1}{3!} - \frac{b_1}{2!}\right] - \dots$$

To satisfy this equation we have to set the coefficients on the L.H.S. to zero and see what happens to R.H.S.

L.H.S. : $a_1 + a_0 = 0$ ①

$a_1 - b_1 - b_0 = 0$ ②

R.H.S. : $\frac{a_1}{2} - b_1 = c_2$ ③ coefficient in front of u_0'', and h^2

$\frac{a_1}{6} - \frac{b_1}{2} = c_3$ ④ coefficient in front of u_0''', and h^3

Choices:

F.E. $\begin{cases} a_0 = -1 & a_1 = 1 \\ b_1 = 0 & b_0 = 1 \\ c_2 = \frac{1}{2} \end{cases}$ ① satisfied ② satisfied

B.E. $\begin{cases} a_0 = -1 & a_1 = 1 \\ b_1 = 1 & b_0 = 0 \\ c_2 = -\frac{1}{2} \end{cases}$

$C_{p+1} \neq 0$ $\boxed{\text{Truncation Error}}$

$\boxed{p \text{ is order of integration}}$

T.R. $a_1 = 1 \quad a_0 = -1$ ① and ②

$b_1 = b_0 = 1/2$ satisfied.

but now $C_2 = 0$!

$$C_3 = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12} \neq 0$$

$\therefore$ truncation error $= 3$

order of integration $= 2$.

Stability of Integration

consider the model equation:

$$\begin{cases} u' = \lambda u & \text{(O.D.E.)} \\ u(0) = u_0 & \text{(I.C.)} \end{cases}$$

Sol^{n} is $\quad u(t) = u_0 e^{\lambda t}$

For a system of equations $\underline{u}' = A\underline{u}$ it will be the eigenvalues of A that will give the exponents λ, so it is sufficient to study the above scalar model equation.

λ may be real or complex, although we'll only be interested in real u.

If we apply F.E.:

$$u_1 = (1+\lambda h)u_0$$
$$u_2 = (1+\lambda h)^2 u_0$$
$$\vdots$$
$$u_n = (1+\lambda h)^n u_0$$

For a stable O.D.E. $Re(\lambda) < 0$ and the sol^{n} is bounded.
For the numerical solution to also remain bounded requires

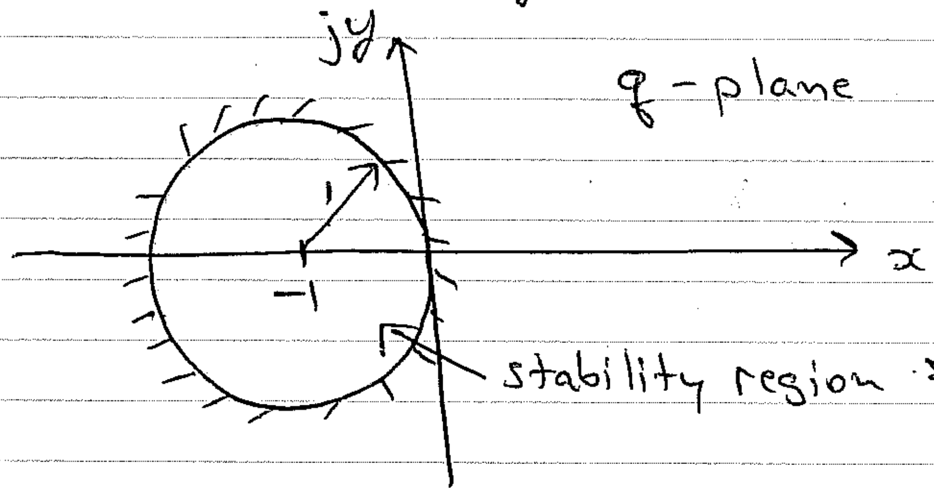
$$\boxed{|1 + h\lambda| \leq 1}$$ Stability condition for F.E.

let $h\lambda = q = x + jy$

$\therefore$ stability is maintained if

$$|1 + x + jy| \leq 1$$

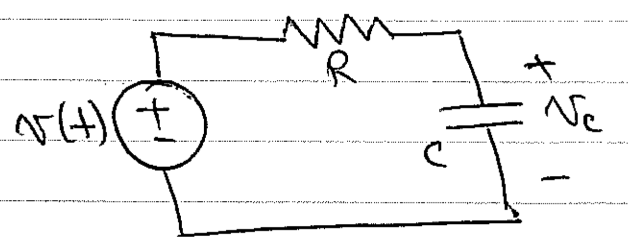
or $$(1+x)^2 + y^2 \leq 1$$



$\therefore$ we must choose h, the time step, such that $q = \lambda h$ is inside the circle.

if $|\lambda|$ is large, h will be small

Ex:



$$C \frac{dv_c}{dt} = \frac{v(t) - v_c(t)}{R}$$

$$\dot{v}_c = \frac{-1}{RC} v_c + \frac{v(t)}{RC}$$

$\therefore$ $\lambda = \frac{-1}{RC}$ (real).

$\therefore$ $\frac{h}{RC} < 2$ or $\boxed{h < RC.}$

ex: $R = 1\,k\Omega$ $C = 1\,\mu F$ $h < 10^{-3}$

Backward Euler:

$$u_1 = u_0 + h u_1' = u_0 + h \lambda u_1$$

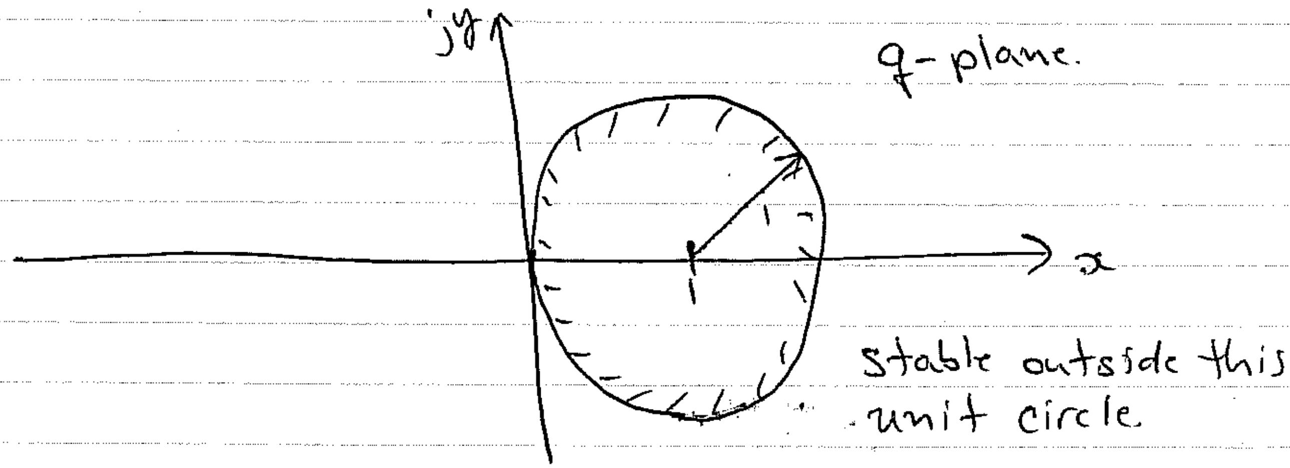
$$u_1 = \frac{u_0}{1-h\lambda} = \frac{u_0}{1-q}$$

$$u_n = \left[\frac{1}{1-q}\right]^n u_0$$

amplification factor is $\left(\frac{1}{1-q}\right)^n$

$\therefore$ we require $\left|\frac{1}{1-q}\right| \le 1$

$\therefore \quad (1-x)^2 + y^2 \ge 1$



Because a stable O.D.E. requires λ in Left half plane, $h\lambda = q$ will always be stable for any h for a stable O.D.E.

B.E. will give a stable update scheme even for unstable O.D.E.'s !

Trapezoidal Formula :

$$u_1 = u_0 + \frac{\lambda h}{2}(u_1 + u_0)$$

$$u_1 = \left(\frac{1 + \lambda h/2}{1 - \lambda h/2}\right) u_0$$

$$u_n = \left(\frac{2+q}{2-q}\right)^n u_0$$

amplification factor is $\left(\frac{2+q}{2-q}\right)^n$

$\therefore$ For stability : $\left|\frac{2+q}{2-q}\right| \le 1$

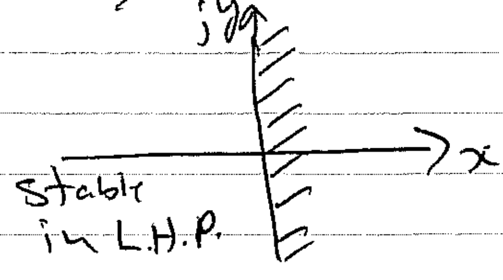
$$\left|\frac{2+x+jy}{2-x-jy}\right| \le 1$$

$$(2+x)^2 + \cancel{y^2} \le (2-x)^2 + \cancel{y^2}$$

$$\cancel{4} + 4x + \cancel{x^2} \le \cancel{4} - 4x + \cancel{x^2}$$

$$8x < 0$$

$$\boxed{x < 0}$$



$x = h\,Re(\lambda)$

$\therefore$ T.R. is stable for all stable O.D.E.'s, and unstable for all unstable O.D.E.'s

This is a nice property.

Also the order of integration is better.

A-stable : if an integration formula leads to a bounded solution for the model O.D.E. $u' = \lambda u$ for any h and any n (if $Re(\lambda) < 0$) then it is called A-stable.

Stiff System

- contains some λ that are -ve but very large magnitude

$\Rightarrow$ fast transients.

if we use F.E. h will be very small, but after the transient dies out we would like to switch to a larger h.

$\therefore$ B.E. is prefered (or. T.R.)

A Note on MNA Formulations

when we use MNA to formulate network equations, the method is general and simple to apply, but the formulation results in a system of Algebraic - Differential Equations

$$sC\underline{x} + G\underline{x} = \underline{w}$$

$$\Downarrow$$

$$C\dot{\underline{x}} = -G\underline{x} + \underline{w}$$

the matrix C may be very sparse and non-invertible. This would mean that we couldn't write the system as a pure system of O.D.E.'s

$$\dot{\underline{x}} = A\underline{x} + \underline{w}$$

(Note: state-space formulations do just that.)

This shows up as a problem, for example, when trying to apply Forward Euler to MNA formulations:

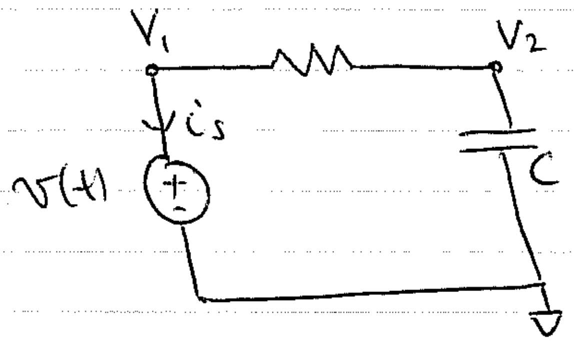
$$h\underline{x}_n' = \underline{x}_{n+1} - \underline{x}_n$$

$$\therefore \quad hC\underline{x}_n' = C\underline{x}_{n+1} - C\underline{x}_n = -hCG\underline{x}_n + hC\underline{w}_n$$

$$C\underline{x}_{n+1} = [C - hCG]\underline{x}_n + hC\underline{w}_n$$

but because C can't be inverted, we can't write an explicit scheme for $\underline{x}_{n+1}$

Example:



MNA gives:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} V_1 \\ V_2 \\ i_s \end{pmatrix} = \begin{pmatrix} -G & G & -1 \\ G & -G & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ i_s \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v(t) \end{pmatrix}$$

but this is really only one O.D.E:

$$C \frac{dV_2}{dt} = GV_1 - GV_2$$

and two algebraic equations:

$$-GV_1 + GV_2 - i_s = 0$$
$$V_1 = v(t)$$

$\therefore$ O.D.E becomes $C \dfrac{dV_2}{dt} = -GV_2 + Gv(t)$

$$\boxed{\frac{dV_2}{dt} = -\frac{G}{C} V_2 + \frac{G}{C} v(t)}$$

F.E. could be applied to this equation as:

$$V_2^{n+1} = V_2^n - \frac{hG}{C} V_2^n + \frac{hG}{C} v(nh)$$

So in order to use F.E. we have to eliminate all the algebraic equations, and convert the system of algebraic-differential equations to a system of pure differential equations.

Notice that B.E. or T.R. don't have trouble with C being singular in

$$C \dot{\underline{x}} = -G \underline{x} + \underline{w}$$

<u>B.E.</u>

$$C(\underline{x}_{n+1} - \underline{x}_n) = -hG \underline{x}_{n+1} + h\underline{w}_{n+1}$$

$$\boxed{(C + hG)\underline{x}_{n+1} = C\underline{x}_n + h\underline{w}_{n+1}}$$

so as long as $C+hG$ is not singular, we're okay.

T.R.

$$C(\underline{x}_{n+1} - \underline{x}_n) = -\frac{h}{2} G(\underline{x}_{n+1} + \underline{x}_n) + \frac{h}{2}(\underline{w}_{n+1} + \underline{w}_n)$$

$$\boxed{\left(C + \frac{h}{2} G\right) \underline{x}_{n+1} = \left(C - \frac{h}{2} G\right) \underline{x}_n + \frac{h}{2}(\underline{w}_{n+1} + \underline{w}_n)}$$